

Bessel Polynomial Expansions in Spaces of Holomorphic Functions

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General methods of nuclear Fréchet spaces in conjunction with the theory of Köthe sequence spaces have been used to obtain a basis criterion for power sequence spaces. How this basis criterion is thus yielding a refinement of Cannon's criterion in the case of space of holomorphic functions is illustrated here. Hence, we give a characterization of those infinite matrices for which the respective Bessel polynomials constitute a basis in the space of holomorphic functions. Consequently, an answer to the question posed by H. L. Krall and O. Frink (*Trans. Amer. Math. Soc.* **65**, 1949, 100–115) concerning the actual expansion of a holomorphic function in terms of Bessel polynomials is given. Also, we append a new criterion for basis transforms in such a space of holomorphic functions by means of order of magnitude of coefficients of polynomials. © 1998 Academic Press

INTRODUCTION

Since we are interested in basis transforms it will be convenient always to look upon the space together with a basis, and notations of the form (V, b) will be used, where V is the vector space and $b = (b_n)_{n=1}^\infty$ is a topological basis for V . If V and W are isomorphic and there exists an isomorphism ϕ , called a basis preserving isomorphism mapping the basis b of V to a basis c of W (i.e., $\phi(b_n) = c_n$ for all n), (V, b) and (W, c) are called similar. This relation is written $(V, b) \simeq (W, C)$. If y is a basis for V ,

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then each $x_n \in V$ has a unique representation of the form

$$x_n = \sum_{j \in \mathbb{N}} Q_{n,j} y_j \quad (\text{A})$$

so that formally

$$y_k = \sum_{n \in \mathbb{N}} P_{k,n} \left(\sum_{j \in \mathbb{N}} Q_{n,j} y_j \right). \quad (\text{B})$$

Let E be an NF-space (nuclear Fréchet space) with basis $(x_n)_{n \in \mathbb{N}}$ and let $(y_n)_{n \in \mathbb{N}}$ be a sequence in E satisfying the relations (A) and (B). The question arises as to which conditions upon the matrices P and Q may we conclude that $(y_n)_{n \in \mathbb{N}}$ is a basis for E . In other words P defines a basis transform in E .

Let $\alpha = (\alpha_n)_{n \in \mathbb{N}}$ be an increasing sequence of positive numbers and put $0 < R < \infty$ fixed,

$$a_n^k = (Re^{-1/k})^{\alpha_n}.$$

Then calling $A = (a_n^k)$ and associating with it the Köthe sequence space $K(A)$, it is well known that

$$K(A) \text{ is nuclear} \Leftrightarrow \lim_{n \rightarrow \infty} \frac{n \log n}{\alpha_n} = 0.$$

In the case $R = \infty$, we put

$$a_n^k = e^{k\alpha_n}$$

and $A = (a_n^k)$. It is well known that

$$K(A) \text{ is nuclear} \Leftrightarrow \limsup_{n \rightarrow \infty} \frac{\log n}{\alpha_n} < +\infty.$$

In the case of nuclear power series spaces the Köthe sequence space $(K(A), e)$, where e is a basis given by $e_k = (\delta_{nk})$, is denoted by $K(\alpha, R)$. In [2, Theorem 3.1] we have established which NF-spaces with basis are similar to a power sequence space $K(\alpha, R)$ and have shown that for $0 < R < +\infty$, $K(\alpha, R) \simeq (\mathcal{O}(B(R)), \zeta)$, the space of functions holomorphic in $B(R)$ can be found with the classical basis of monomials and $\alpha = (\alpha_n)_{n \in \mathbb{N}}$ with $\alpha_n = n$ for all $n \in \mathbb{N}$. It follows that the question posed above can be answered for the case of a power sequence space by the following theorem [2, Theorem 3.2].

THEOREM A. *Let $K(\alpha, R)$ be a power sequence space with basis e and let P be an infinite matrix. Then Pe is a basis for $K(\alpha, R)$ if and only if*

(C.1) *there exists an infinite matrix Q such that $PQ = QP = I$*

(C.2) *for each $m \in \mathbb{N}$, there exists $l \in \mathbb{N}$ and $K_m > 0$ such that for all $n, k, t \in \mathbb{N}$,*

$$|Q_{n,k}| |P_{k,t}| r_m^{\alpha_t} \leq K_m r_l^{\alpha_n}.$$

In the present paper we illustrate how this basis criterion leads to a refinement of Cannon's criterion in the space $(\mathcal{O}(B(R)), \zeta)$. This modified criterion is named in [2], a basis transform in $K(\alpha, R)$ and is used here to give a characterization of those infinite matrices for which the respective Bessel polynomials constitute a basis in $(\mathcal{O}(B(R)), \zeta)$. A new criterion for basis transforms in $(\mathcal{O}(B(R)), \zeta)$ by means of order of magnitude of coefficients of the polynomials is introduced as well.

1. NOTATIONS AND TERMINOLOGY

Let ω be the space of all infinite sequences of complex numbers. If $A = (a_n^k)$ is a matrix of real nonnegative numbers such that $a_n^k \leq a_n^{k+1}$ for all n and k and for each n there exists a k such that $a_n^k \neq 0$, then

$$K(A) = \left\{ x = (x_n) \in \omega : \|x\|_k = \sum_n a_n^k |x_n| < +\infty \right\}$$

is called a Köthe sequence space with the topology given by the seminorms $\|\cdot\|_k$. It is well known that if $K(A)$ is nuclear, then e is a basis for $K(A)$, where e is given by $e_k = (\delta_{nk})$ and that moreover $(K(A), e) \simeq (S(A), e)$ where $S(A)$ is the supremum space associated with A given by

$$S(A) = \left\{ x = (x_n) \in \omega : \|x\|_k = \sup_n a_n^k |x_n| < +\infty \right\}$$

with the topology defined by the seminorms $|\cdot|_k$. It is well known that if (V, x) is an NF-space and the topology is given by the seminorms p_k , then $(V, x) \simeq (K(A), e)$ where A is given by $a_n^k = p_k(x_n)$. This makes it possible to use alternatively the seminorms p_k , $\|\phi(\cdot)\|_k$, or $|\phi(\cdot)|_k$ where ϕ is the basis preserving isomorphisms. A class of a Köthe sequence space which will be of special interest to us is that of the so-called power series spaces. In the case of nuclear power series spaces the space $(K(A), e)$ is denoted by $K(\alpha, R)$ and the associated supremum space by $S(\alpha, R)$.

Now let (y_n, y'_n) be a complete biorthogonal sequence for V ; that is, the finite linear combinations of the elements y_n are dense in V and $y'_n(y_k) = \delta_{n,k}$ for all n and k . The question arises as to which conditions will ensure that a given complete biorthogonal sequence already constitutes a basis for V . Theorems due to Haslinger [5, 6] have answered this question. In [2] a criteria is given under which a sequence of vectors y in an NF-space E with basis x is itself a basis for E . It is assumed that y is expressed in terms of x by the infinite matrix P , i.e., $y = Px$ ($y_k = \sum_{n \in \mathbb{N}} P_{k,n} x_n, k \in \mathbb{N}$). x' is again the sequence in V' of coordinate functions with respect to x . The existence of an inverse of P in the sense of infinite matrices is determined by the following result of [2]: Suppose that $y = (y_k)_{k \in \mathbb{N}}$ is a basis for E . Then P has a two-sided inverse Q .

2. STATEMENTS AND DISCUSSION

Remarks. (1) It has been shown in [2] that formula (C.2) can be replaced by

$$(C'.2) \quad |Q_{n,k}| |P_{k,t}| \|x_t\|_m \leq K_m \|x_n\|_t \text{ or}$$

$$(C''.2) \quad \sum_{k \in \mathbb{N}} \sum_{t \in \mathbb{N}} |Q_{n,k}| |P_{k,t}| \|x_t\|_m \leq K_m \|x_n\|_l.$$

(2) Let us comment upon the condition (C.1) of the previous criteria (Theorem A). It requires that the infinite matrix P has two-sided inverse Q . Of course, if P is taken to belong to some class of infinite matrices which forms a ring \mathcal{S} under the classical rules of additional multiplication for matrices and moreover $I \in \mathcal{S}$, then, if P has a two-sided inverse $Q \in \mathcal{S}$, Q is unique.

An example of such a ring \mathcal{S} is the ring \mathcal{S}_{up} of all infinite upper-triangular matrices or the ring \mathcal{S}_{lo} of all infinite lower-triangular matrices. It is clear that if, e.g., $P \in \mathcal{S}_{up}$ has all diagonal elements different from zero, then P has a two-sided inverse Q in \mathcal{S}_{up} , which is therefore unique (see [3]).

Nevertheless, it may happen that, although the matrix P we are considering belongs to some ring \mathcal{S} of infinite matrices, P admits a two-sided inverse which does not belong to this ring \mathcal{S} . To illustrate this phenomenon, take for \mathcal{S} the ring of all infinite row-finite matrices and let $P \in \mathcal{S}$ be given by $P = (P_{i,j})_{i,j=1}^{\infty}$ where $P_{i,i} = 1$ for all $i \in \mathbb{N}$, $P_{i,i+1} = -1$ for all $i \in \mathbb{N}$, and $P_{i,j} = 0$ if $i \neq j$ or $j \neq i+1$. Then P admits the two-sided inverse $Q = (Q_{i,j})_{i,j=1}^{\infty}$ with $Q_{i,j} = 0$ for $j < i$ and $Q_{i,j} = 1$ for $j \geq i$. Clearly Q is not a row-finite matrix.

EXAMPLE. Applying the previous matrix P to the space of holomorphic functions $(\mathcal{O}(B(\frac{1}{2})), \zeta)$, $P\zeta$ is given by

$$p_k(z) = z^k - z^{k+1}, \quad k = 0, 1, 2, \dots,$$

and using Theorem A for a basis transform, $p = P\zeta$ is indeed a basis for $\mathcal{O}(B(\frac{1}{2}))$.

Basis Transforms in Power Sequence Spaces

DEFINITION. Let P be an infinite matrix which has a two-sided inverse Q and let $r > 0$ be fixed. Then for $n \in \mathbb{N}$ we put

$$J_n(P, r) = \sup_{k, t} |Q_{n, k}| |P_{k, t}| r^{\alpha_t},$$

$$J(P, r) = \lim_{n \rightarrow \infty} (J_n(P, r))^{1/\alpha_n}$$

if all $J_n(P, r)$ are finite, and $+\infty$ otherwise.

From Theorem A(C.2) we have

THEOREM B. Let $K(\alpha, R)$ be a power sequence space and let P be an infinite matrix with two-sided inverse Q . Then P determines a basis transform in $K(\alpha, R)$ if and only if for all $0 < r < R$

$$J(P, r) < R.$$

From the equivalence of (C'.2) and (C''.2), given $r > 0$, we put

$$Z_n(P, r) = \sum_k \sum_t |Q_{n, k}| |P_{k, t}| r^{\alpha_t},$$

$$Z(P, r) = \lim_{n \rightarrow \infty} (Z_n(P, r))^{1/\alpha_n}.$$

We thus have

THEOREM C. Let $K(\alpha, R)$ be a power sequence space and let P be an infinite matrix having two-sided inverse Q . Then P determines a basis transform in $K(\alpha, R)$ if and only if for all $0 < r < R$.

$$Z(P, r) < R.$$

3. CANNON'S CRITERION REVISITED

Consider the space $(\mathcal{O}(B(R)), \zeta)$ of holomorphic functions. Then Cannon and Whittaker [11] considered infinite matrices P of the following

type:

- (i) P is row-finite
- (ii) P possess a row-finite two-sided inverse Q .

For each $0 < r < R$ the following entities were introduced:

$$\lambda_n(P, r) = \sum_{k, s} |Q_{n, k}| \|P_k\|_r;$$

$$F_n(P, r) = \sup_{t_1, t_2} \sup_{|z| \leq r} \left| \sum_{k=t_1}^{t_2} Q_{n, k} \left(\sum_s P_{k, s} z^s \right) \right|$$

$$\lambda(P, r) = \limsup_{n \rightarrow \infty} \{\lambda_n(P, r)\}^{1/n}$$

$$F(P, r) = \limsup_{n \rightarrow \infty} \{F_n(P, r)\}^{1/n}.$$

DEFINITION. A basis of polynomials $P_j(z)$, $j = 1, 2, \dots$, $z \in \mathbb{C}$, is called an effective basis in a ball $B(R)$ if every function $f(z)$ holomorphic in this ball can be represented in a series $f(z) = \sum_j c_j P_j(z)$; $c_j \in \mathbb{C}$.

Cannon [1] proved that the basis of polynomials determined by $P\zeta$ is effective in $B(R)$ if and only if $F(P, r) < R$ for all $0 < r < R$, and that for the Cannon basis $\lambda(P, r) = F(P, r)$. When we look at $J_n(P, r)$ in the case under consideration, it is clear that for all $n \in \mathbb{N}$ and $0 < r < R$,

$$J_n(P, r) \leq F_n(P, r) \leq \lambda_n(P, r)$$

$$J(P, r) \leq F(P, r) \leq \lambda(P, r).$$

Besides the fact that Cannon only proved the effectiveness of the basis $P\zeta$ under the condition $F(P, r) < R$ for all $0 < r < R$, while we have that $J(P, r) < R$ for all $0 < r < R$ implies that $P\zeta$ is a basis, it obviously follows from $\lambda(P, r) < R$ for all $0 < r < R$ that $J(P, r) < R$ for all $0 < r < R$. Hence our condition using $J(P, r)$ is weaker than the one obtained by Cannon using $\lambda(P, r)$ in the case of so-called Cannon bases (i.e., bases for which $\lambda(P, r) = F(P, r)$). There is even more. Since we can also use $Z(P, r)$ to establish whether or not $P\zeta$ is a basis for $\mathcal{O}(B(R))$ and since clearly for all $n \in \mathbb{N}$ and $0 < r < R$

$$J_n(P, r) \leq F_n(P, r) \leq \lambda_n(P, r) \leq Z_n(P, r)$$

whence

$$J(P, r) \leq F(P, r) \leq \lambda(P, r) \leq Z(P, r),$$

we may conclude that $\lambda(P, r)$ may also be used in the case of non-Cannon bases (general bases). Of course it should also be stressed that the matrices P we are considering need not be row-finite.

Let us illustrate the previous observations in the case of $(\mathcal{O}(B(1)), \zeta)$.

EXAMPLE. Consider the function $1/(1 - z) = \sum_0^\infty z^j \in \mathcal{O}(B(1))$. Then we claim that the set

$$\left\{ \frac{1}{1 - z}, z, z^2, \dots, z^n, \dots \right\}$$

is a basis for $\mathcal{O}(B(1))$.

Indeed, the associated infinite matrix P is given by $P_{i,i} = 1$, $P_{1,j} = 1$, $j \in \mathbb{N}$, $P_{i,j} = 0$, $i \neq j$; it is an upper triangular and has diagonal elements 1. Hence it possess a two-sided inverse Q ; it is given by $Q_{i,i} = 1$, $Q_{i,j} = -1$, $j > i$, $Q_{i,j} = 0$, $i < j$.

Taking $\alpha = (\alpha_n) = \mathbb{N}$, a straightforward calculation yields for each $0 < r < 1$ that

$$(i) \quad J_0(P, r) = \sup_{t \in \mathbb{N}, t \geq 1} r^t = 1,$$

$$(ii) \quad J_n(P, r) = r^n, \quad n \geq 1.$$

Consequently, $J(P, r) = r$. So that $J(P, r) < 1$ for each $0 < r < 1$. Hence $P\zeta$ is a basis for $\mathcal{O}(B(1))$.

4. BASIS OF BESSEL POLYNOMIALS

Let $\mathcal{O}(B(R))$ be the space of holomorphic functions on the disc $B(R) = \{z : |z| < R\}$ ($0 < R \leq \infty$) endowed with the topology of uniform convergence on compact subsets of $B(R)$. The space $\mathcal{O}(B(R))$ is a nuclear Fréchet space (NF-space) (see [10]), the topology of which can be defined by the system of norms $\|f\|_r = \max_{|z|=r} |f(z)|$ ($0 < r < R$, $f \in \mathcal{O}(B(R))$). Suppose that $P = (P_{i,j})$ and $Q = (Q_{i,j})$ are infinite matrices, and the infinite identity matrix will be denoted by I .

A class of orthogonal polynomials which arises in the theory of traveling spherical waves is called Bessel polynomials because of its entimate relationship with the Bessel functions of half integral order. According to Krall and Frink [8] the sequence $(P_n(z))_{n=0}^\infty$ of Bessel polynomials is given by

$$P_0(z) = 1; \quad P_n(z) = \sum_{k=0}^n \frac{(n+k)!}{2^k k! (n-k)!} z^k, \quad n \geq 1. \quad (1)$$

In the sequel we consider the matrix Q as the inverse of P . We want to find a necessary and sufficient condition on the matrices P and Q such that each function $f \in \mathcal{O}(B(R))$ has a unique expansion

$$f(z) = \sum_{k=0}^{\infty} c_n P_n(z)$$

which is convergent in the topology of $\mathcal{O}(B(R))$. In other words, we characterize those matrices Q and P for which the respective Bessel polynomials constitute a basis in $\mathcal{O}(B(R))$. There are some results in this direction (see [8, 9]). In [9] Nassif uses the approach of Whittaker and Cannon [11]. Our approach is quite different from the methods of Whittaker and Cannon [11] and Krall and Frink [8], because we use such methods of pure functional analysis as biorthogonal sequences, nuclearity, bases, Köthe sequence spaces, and basis transforms in power sequence spaces.

THEOREM 1. *The sequence of Bessel polynomials $(P_n(z))_{n=0}^{\infty}$ constitutes a basis in the space of holomorphic functions $\mathcal{O}(B(R))$, $0 < R < \infty$.*

The following theorem gives an answer to the question posed by Krall and Frink [8] concerning the actual expansion of a holomorphic function in terms of Bessel polynomials.

THEOREM 2. *Any function $f(z)$ holomorphic in $B(R)$, where $R \geq 0$, can be expanded in a series of Bessel polynomials of the form $f(z) = \sum_{k=0}^{\infty} c_n P_n(z)$, where*

$$c_n = 2^n (2n+1) \sum_{j=0}^{\infty} (-2)^j f(0)^{(n+j)} \bigg/ [j!(2n+j+1)]. \quad (2)$$

Proof of Theorem 1. From the formula of Bessel polynomials (1) we have the matrix P such that

$$P_{n,k} = 2^{-k} (n+k)! / (k!(n-k)!), \quad 0 \leq k \leq n; n \geq 1, \quad (3)$$

Since $(z^n)_{n=0}^{\infty}$ is a basis for $\mathcal{O}(B(R))$, each z^n admits the representation

$$z^n = \sum_i Q_{n,i} P_i(z), \quad (4)$$

so that the matrix $Q = (Q_{n,i})$ is the unique inverse of the matrix $P = (P_{n,i})$ (see [2, Theorem 2.1]).

Carrying out the product $PQ = I$, we obtain

$$P_{i,i} Q_{i,i} = 1, \quad (5)$$

and

$$\sum_{k=i}^n P_{n,k} \pi_{k,i} = 0, \quad n > i, i \geq 0. \quad (6)$$

Setting $n = m + i$ in (6) and inserting the values of $(P_{n,k})$ from (3) we obtain

$$Q_{m+i,i} = - \sum_{s=0}^{m-1} \frac{2^{m-s} (2i + m + s)! (m + i)!}{(i + s)! (m - s)! (2i + 2m)!} Q_{i+s,i}. \quad (7)$$

Applying (7) and the identity

$$\sum_{s=0}^m (-1)^s \binom{m}{s} \binom{k + m + s}{m - 1} = 0$$

(this is in fact the coefficient of x^{k+m+1} in the expansion of $(1 + x)^m (1 + x)^{-m} = 1$), we can easily deduce by induction that

$$Q_{m+i,i} = \frac{(-2)^m (m + i)! (2i + 1)!}{m! i! (2i + m + 1)!} Q_{i,i}, \quad m, i \geq 0.$$

Substituting for $Q_{i,i}$ from (5) it follows that

$$Q_{n,i} = \frac{(-1)^{n-i} 2^n n! (2i + 1)!}{(n - i)! (n + i + 1)!}, \quad 0 \leq i \leq n. \quad (8)$$

Now forming the sum $Z_n(P, r)$ with $\alpha_n = n$, $n \in \mathbb{N}$ and apply (3) and (8) we obtain after certain reduction

$$\begin{aligned} Z_n(P, r) &= \sum_{k=0}^n \sum_{j=0}^k \frac{2^n n! (2k + 1)!}{(n - k)! (n + k + 1)!} \cdot \frac{(k + j)!}{2^j j! (k - j)!} r^j \\ &= \sum_{j=0}^n \sum_{k=j}^n \frac{2^n n! (2k + 1)!}{(n - k)! (n + k + 1)!} \cdot \frac{(k + j)!}{2^j j! (k - j)!} r^j \\ &= \sum_{j=0}^n \sum_{l=0}^{n-j} \frac{2^n n! (2l + 2j + 1)!}{(n - l - j)! (n + l + j + 1)!} \cdot \frac{(l + 2j)!}{2^j j! l!} r^j \\ &= 2^n \sum_{j=0}^n \binom{n}{j} \left(\frac{r}{2} \right)^j \sum_{l=0}^{n-j} \binom{n-j}{l} \frac{(2l + 2j + 1)(2j + l)!}{(n + j + l + 1)!}. \end{aligned}$$

Introducing the summation index, $q = n - j$, and in view of $2l + 2j + 1 \leq 2n + 1$,

$$\begin{aligned} Z_n(P, r) &< (2n + 1)r^n 2^n \sum_{q=0}^n \frac{2^{q-n}}{r^q} \binom{n}{q} \cdot \sum_{l=0}^q \binom{q}{l} \frac{(2n - 2q + l)!}{(2n - q + l + 1)!} \\ &< (2n + 1)r^n \sum_{q=0}^n \frac{2^q}{r^q} \binom{n}{q} \cdot 2^q \frac{1}{q!} \\ &= (2n + 1)r^n \sum_{q=0}^n \binom{n}{q} \left(\frac{4}{r}\right)^q / q! \\ &= (2n + 1)r^n B_n, \quad \text{say.} \end{aligned} \quad (9)$$

Effecting the transformation $b = a(1 + a)^{-1}$ on the function $a \exp(4a/r) = \sum_{n=0}^{\infty} (4/r)^n a^{n+1}/n!$ it follows from (9) that

$$H(b) = \left(\frac{b}{1-b}\right) \exp\left\{\frac{4b}{r(1-b)}\right\} = \sum_{n=0}^{\infty} B_n b^{n+1}.$$

This function is regular in $|b| < 1$; hence by Cauchy's inequality we have

$$B_n < K/\gamma^{n+1} \quad (0 < \gamma < 1),$$

where $K = \max_{|b|=\gamma} |H(b)| < \infty$. Inserting this in (9), making n tend to infinity, and in view of Theorem B we obtain

$$Z(P, r) \leq \frac{r}{\gamma}.$$

According to [2, Theorem 3.2 and Theorem 3.4] we infer that $(P_n(z))_{n=0}^{\infty}$ constitutes a basis in $\mathcal{O}(B(r))$, $r > 0$. In other words, the matrix P determines a basis transform in $\mathcal{O}(B(r))$. Thus Theorem 1 follows. ■

Proof of Theorem 2. Suppose that $f(z)$ is any function holomorphic in $B(R)$, $R > 0$. Then substitute for z^n from (4) in the Taylor expansion $\sum (f^{(n)}(0)/n!)z^n$ of $f(z)$ about the origin to get formally the series $\sum c_n P_n(z)$, where

$$c_n = \sum_{s=0}^{\infty} Q_{n+s,n} f^{(n+s)}(0) / (n+s)!.$$

Hence inserting the values of $Q_{n+s,n}$ from (8), Theorem 2 follows. ■

An important subclass of the sequence of polynomials $(y_k(z))_{k \in \mathbb{N}}$ is the class for which

$$D_n = O(n),$$

as n tends to infinity, where D_n is the largest degree of any $y_k(z)$ in the representation (4).

For such subclass the following property can be established.

LEMMA 1. *If $(P_n(z))$ ($n \in \mathbb{N}$) is a sequence of polynomials such that $D_n = O(n)$ and $Z(P, 0+) < \infty$, then we have*

$$\frac{Z(P, R)}{R^\beta} \leq \frac{Z(P, r)}{r^\beta}, \quad \text{for } 0 < r \leq R < \infty, \quad (10)$$

where $\beta = \lim_{n \rightarrow \infty} (D_n/n)$.

Proof. One can use the same argument of Cannon's result [1] for $\lambda(r)$ but for $Z(P, r)$ and this lemma follows. ■

For the case of the sequence of Bessel polynomials $(P_n(z))_{n=0}^\infty$ we have $\beta = 1$. Then the following proposition is an immediate consequence of Lemma 1 and Theorem 3.4 of [2].

PROPOSITION 1. *If the sequence of Bessel polynomials $(P_n(z))_{n=0}^\infty$ constitutes a basis in a space $\mathcal{O}(B(r))$ ($r < \infty$), then it constitutes a basis in any space $\mathcal{O}(B(R))$ ($R > r$).*

5. A NEW CRITERION FOR POLYNOMIAL BASES

Let $(P_n(z))_{n=0}^\infty$ be a sequence of polynomials of degree n such that $P_0 = P_{0,0} \neq 0$ and set

$$P_n(z) = \sum_{k=0}^n P_{n,k} z^k,$$

where $P_{n,n} \neq 0$ for each $n \in \mathbb{N}$. Such a sequence of polynomials is always complete in $\mathcal{O}(B(R))$ ($0 < R \leq \infty$) since the functions $(z^n)_{n=0}^\infty$ are representable as finite linear combinations of the polynomials P_n . If we consider the space $\mathcal{O}(B(R))$ as a Köthe sequence space $K(A)$, then the polynomials P_n correspond to the sequences $(P_{n,k})_{k=0}^\infty$, where we set $P_{n,k} = 0$ for $k > n$.

The sequence of biorthogonal linear functionals $L_n = (L_{n,k})_{k=0}^\infty \in K'(A)$ for the sequences $(P_{n,k})_{k=0}^\infty \in K(A)$ has the property that $L_{n,k} = 0$ for $k < n$.

Our new criterion for basis transforms in $(\mathcal{O}(B(R)), \zeta)$ is formulated in the following theorem.

THEOREM 3. *Let $(\mathcal{O}(B(R)), \zeta)$ be the space of holomorphic functions and let $(P_n(z))_{n=0}^{\infty}$ be a sequence of polynomials of degree n such that $P_{n,n} = 1$. The matrix $P = (P_{n,k})$ of coefficients in $(P_n(z))_{n=0}^{\infty}$ determines a basis transform in $(\mathcal{O}(B(R)), \zeta)$ for $R \geq (1 + M)$ if and only if*

$$|P_{n,k}| \leq M, \quad 0 \leq k \leq n-1. \quad (*)$$

Proof. Let $K(\alpha, R)$ be a power sequence space. It is supposed that $(P_n(z))_{n=0}^{\infty}$ allows representation of the form (A) and (B).

Since $P_n(z)$ is of degree n , then Theorem 2.1 of [2] immediately yields that there exists a matrix Q such that $PQ = QP = I$, where P and Q are lower-semi-matrices and P is the matrix of coefficients in $(P_n)_{n=0}^{\infty}$.

Since the coefficient of the highest power in $(P_n(z))_{n=0}^{\infty}$ is taken equal to one, then using Lemma 106 of [12], it will be found that

$$|Q_{n,k}| \leq M(1 + M)^{n-k-1} < (1 + M)^{n-k}.$$

In view of the arguments used in the proof of Theorem 2.2 and Theorem 3.2 of [2] then given $r \geq 1 + M$ one can obtain

$$\begin{aligned} Z_n(P, r) &= \sum_{k=0}^{n-1} \sum_t^{k-1} |Q_{n,k}| |P_{k,t}| r^t + r^n \\ &\leq \sum_{k=0}^{n-1} \sum_t^{k-1} (1 + M)^{n-k} M r^t + r^n \\ &< \sum_{k=0}^{n-1} (1 + M)^{n-k} (k + 1) r^k + r^n \\ &< \sum_{k=0}^{n-1} (k + 1) r^n + r^n \\ &< \frac{1}{2}(n + 1)(n + 2) r^n. \end{aligned}$$

For $0 < r < R$ it is easily seen that $Z(P, r) < R$. So, if P is an infinite matrix (in our case a lower-semi-matrix) acting on $\zeta = (z^n)_{n=0}^{\infty}$ then a direct translation of Theorem 2.2 of [2] taking into consideration that $K(\alpha, R) \simeq (\mathcal{O}(B(R)), \zeta)$, $0 < R < +\infty$, and applying Theorem C above, gives the necessity of the condition of the theorem.

To prove its sufficiency, consider the following example due to Whitaker [12].

Set

$$P_0(z) = 1, \quad P_n(z) = z^n + M(z^{n-1} - z^{n-2} + z^{n-3} - \dots) \quad (n \geq 1),$$

then again by Lemma 106 of [12] one can get

$$Q_{n,k} = (-1)^{n-k} M(1+M)^{n-k-1} \quad (k = 0, 1, 2, \dots, n-1).$$

In fact if a function $f(z)$ is taken to be $(1+z+M)^{-1}$, then from the expansion of $f(z)$ in terms of the polynomials $(P_n(z))_{n=0}^\infty$ namely $f(z) \sim \sum_{n=0}^\infty \Delta_n(0) P_n(z)$ we have

$$\Delta_n(0) + \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} Q_{k,n}$$

and

$$\Delta_0(0) = \frac{1}{1+M} + \frac{M}{(1+M)^2} + \frac{M}{(1+M)^2} + \dots,$$

which does not converge. That is to say, there exists a set of polynomials satisfying the condition (*) of the theorem and a function $f(z)$, regular in $|z| < 1+M$, for which the series $\sum_{n=0}^\infty \Delta_n(0) P_n(z)$ does not converge. Therefore the condition $R \geq 1+M$ is essential. This completes the proof of the theorem. ■

Remark. The above theorem gives also that $(P_n)_{n=0}^\infty$ constitutes a basis in the space $(\mathcal{O}(B(R)), \zeta)$ for $R \geq (1+M)$ according to Theorem 3.2 of [2].

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